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Abstract

A new algebraic method for extracting tori from a minimal point set, made of two oriented points and a simple point, is proposed. We prove a degree bound on the number of such tori; this bound is reached on examples, even when we restrict to smooth tori. Our method is based on pre-computed closed formulae well suited for numerical computations with approximate input data.

Keywords: mixed set of 3D points, tori, interpolation method.

1 Introduction

The extraction of geometric primitives from 3D point clouds is an important problem in reverse engineering. These 3D point clouds are typically obtained by means of accurate 3D scanners and there exist several methods for performing the 3D geometric primitives extraction. An important category among them are the statistical and iterative methods based on the RANSAC (RANdom SAmple Consensus) paradigm [3, 7, 8]. Key ingredients in this approach are geometric routines that are capable to produce an instance of a given type of shape from a small number of points. For instance, computing the equation of a plane passing through three given points, or passing through a point-with-normal (a point with a normal vector), are basic routines that are intensively used in RANSAC-based methods. In practice, the most used types of shapes are planes, spheres, cylinders, cones and tori. While devising such routines is relatively straightforward for planes and spheres, the cases of cylinders, cones and tori are much more difficult. In a previous work [1] we proposed a detailed analysis and efficient algorithms for cylinders and cones. In this note we treat the case of tori. More precisely, we provide a new method for extracting tori from the smallest possible number of conditions, that is to say from two point-with-normal and a single point, which account for seven parameters, i.e. the same number as the degrees of freedom of the considered interpolation problem. We emphasize that it is very important to compute shapes from the smallest possible number of conditions in order to guarantee efficiency and accuracy in these interpolation processes. Most of the methods that are currently used in practical applications for cylinders, cones and tori are based on the solving of overdetermined linear systems so that the computed shapes are not interpolating the point-with-normal data but are only approximating them.

In the sequel, an *oriented point* is a couple of a point and a nonzero vector. A surface is said to interpolate an oriented point if the point belongs to the surface and its associated vector is colinear to the normal of the surface at this point. Notice that we are not assuming that the orientation of the normal of the point is the same as the orientation of the surface. Moreover, it is important to deal with inhomogeneous data, that is to say some points are oriented but not all, in order to take into account the estimated accuracy of oriented point clouds that are obtained by means of normal estimation algorithms. A set of data made of points and oriented points is called a mixed set of points.

Previous methods for the extraction of tori from a small number of points in a RANSAC-like approach have been treated from an overdetermined number of conditions, i.e. mixed set of points that are bigger than necessary, which has the consequence that computed tori are not exactly interpolating the data. For instance, in [6] tori are approximated from four oriented points, i.e. twelve conditions, and in [4, 5] tori are approximated from three oriented points, i.e. nine conditions. In this note, we propose a new method that interpolate tori from two oriented points and a single point, i.e. from seven conditions, which is precisely the number of parameters needed to instantiate a torus. At the heart of this contribution is an original and subtle modelisation of this interpolation problem, in opposition to the brute force approach that leads to huge and time consuming treatments of polynomial systems of equations of degree 4 in 7 unknows. Based on this modelisation, our approach relies on adapted algebraic techniques and allows to develop an efficient interpolation algorithm in the context of numerical computations in double precision with approximate data. As a byproduct, we will also get the following theorem of enumerative geometry which seems to be unknown.

Theorem. *There exist at most eight non-degenerated tori (or an infinity) that interpolate three distinct points, two of them being oriented.*

Our strategy of proof relies on geometric constructions related to 3D interpolation of circles which are described in Section 2. Our new torus interpolation method is developed in Section 3. In Section 4, we will report on our numerical experiments, and also show an example for which this upper bound (eight) is reached, even when we restrict to smooth tori.

2 3D circles passing through two oriented points

The spine of a torus is a 3D circle. Such a circle depends on six parameters, namely three parameters for the coordinates of its center, two parameters for its supporting plane and one last parameter corresponding to its radius. Thus 3D circles and pairs of oriented points share the same number of degrees of freedom, namely six, and hence one may ask how many circles pass through two oriented points.

We recall that a vector V is said normal to a 3D curve $\mathcal{C}$ at a smooth point M when V is orthogonal to the tangent vector to $\mathcal{C}$ at M.

Proposition 1. *We suppose that two distinct points A_1 and A_2 and two nonzero vectors N_1 and N_2 are given. For $i = 1, 2$, we denote by Ω_i the intersection point, possibly at infinity, between the bisecting plane of the segment $[A_1A_2]$ and the line passing through A_i and parallel to N_i (see Figure 1). Consider the following fitting problem (P): determine all the 3D circles that interpolate A_1 and A_2 and which are normal to N_1 at A_1 and normal to N_2 at A_2.*

(a) *If $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are not both at infinity, then there is one and only one circle that satisfies (P).*

(b) *If $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are both at infinity, then there is no circle that satisfies (P).*

(c) *If $\Omega_1 = \Omega_2$ then there are infinitely many circles that satisfy (P).*

Proof. First, we observe that a 3D circle $\mathcal{C}$ defines a sheaf of spheres whose centers belong to the line L passing by the center of $\mathcal{C}$ and orthogonal to its supporting plane. Suppose that a point A on $\mathcal{C}$ and a nonzero vector N are given. Denote by D the line through A and parallel to N. Then, N is orthogonal to $\mathcal{C}$ at A if and only if N is orthogonal at A to one and only one of the spheres of the sheaf associated to $\mathcal{C}$, namely the one whose center is the intersection point between the lines D and L. We notice that if D and L are parallel lines, i.e. if N is normal to the supporting plane of $\mathcal{C}$, then the corresponding "limit sphere" of the sheaf of spheres has to be seen as the supporting plane of $\mathcal{C}$, since its center is at infinity and its radius is infinite.

Now, returning to our fitting problem, let $\mathcal{C}$ be a circle that interpolates the two distinct points A_1 and A_2. By the previous observation, we have that the vector N_1 is normal to $\mathcal{C}$ at A_1 if and only if N_1 is normal at A_1 to the sphere S_1 whose center is Ω_1 and that goes through A_1. In other words, the vector N_1 is normal to $\mathcal{C}$ at A_1 if and only if $\mathcal{C}$ is contained in the sphere S_1. Similarly, the vector N_2 is normal to $\mathcal{C}$ at A_2 if and only if $\mathcal{C}$ is contained in the sphere S_2 whose center is Ω_2 and that goes through A_2. We recall that if Ω_1, respectively Ω_2, is at infinity then S_1, respectively S_2, is the normal plane to N_1 through A_1, respectively the normal plane to N_2 through A_2.

To conclude the proof we see that if $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are not both at infinity then the intersection of S_1 and S_2 defines a unique circle because this is the intersection of two spheres, or a sphere and a plane, which contains the two distinct points A_1 and A_2. If $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are both at infinity then S_1 and S_2 are two distinct planes whose intersection is the line through A_1 and A_2, so there is no solution to (P) in this case. Finally, if $\Omega_1 = \Omega_2$ then S_1 and S_2 are both the same sphere, or the same plane, and hence any circle on this sphere, or on this plane, that goes through A_1 and A_2 will give a solution to (P). $\square$

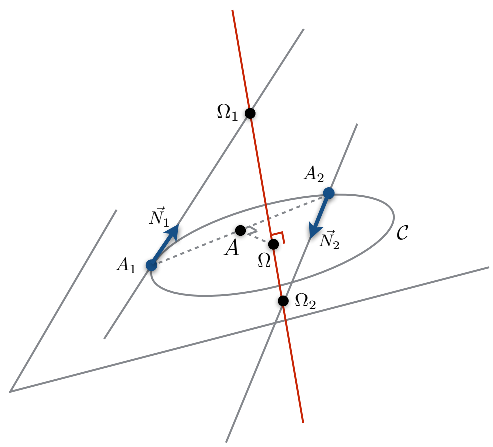

Figure 1: Fitting a 3D circle passing through two distinct points with normal directions.

The above geometric constructions can be quantified to provide explicit formulae that we will use in the next section. Without loss of generality, one may assume that N_1 and N_2 are unitary vectors, which we will do in this section.

Case (a_1)**.** We begin with a sub-case of the first item of Proposition 1, that we denote (a_1): $\Omega_1 \neq \Omega_2$ and neither Ω_1 nor Ω_2 is at infinity. Let $\Omega_1 := A_1 + \lambda_1 N_1$ and $\Omega_2 := A_2 + \lambda_2 N_2$. We have $\|\Omega_1 A_1\|^2 = \|\Omega_1 A_2\|^2$ and $\|\Omega_2 A_1\|^2 = \|\Omega_2 A_2\|^2$. Let A denotes the middle of A_1 and A_2 (see Figure 1). Since the plane (A, Ω_1, Ω_2) is the plane of symmetry of the segment $[A_1, A_2]$, we have $\overrightarrow{\Omega_1 A} \cdot \overrightarrow{A_1 A_2} = 0$ and $\overrightarrow{\Omega_1 \Omega} \cdot \overrightarrow{A_1 A_2} = 0$. We deduce that $\left(\overrightarrow{\Omega_1 A_1} + \frac{1}{2}\overrightarrow{A_1 A_2}\right) \cdot \overrightarrow{A_1 A_2} = 0$ and we get

$$\lambda_1 = \frac{\|A_1 A_2\|^2}{2\overrightarrow{N_1} \cdot \overrightarrow{A_1 A_2}}, \ \lambda_2 = \frac{\|A_1 A_2\|^2}{2\overrightarrow{N_2} \cdot \overrightarrow{A_2 A_1}}. \tag{1}$$

Now, the center Ω of the circle $\mathcal{C}$ satisfies $\|\Omega\Omega_1\|^2 = \|\Omega_1 A_1\|^2 - \|A_1 \Omega\|^2 = \lambda_1^2 - R^2$, and similarly for Ω_2, where R stands for the radius of $\mathcal{C}$. Substituting in the expression of the squared norm of $\overrightarrow{\Omega\Omega_2} = \overrightarrow{\Omega\Omega_1} + \overrightarrow{\Omega_1\Omega_2}$, we obtain $\lambda_2^2 - R^2 = \lambda_1^2 - R^2 + \|\Omega_1\Omega_2\|^2 + 2\overrightarrow{\Omega\Omega_1}.\overrightarrow{\Omega_1\Omega_2}$ and we deduce

$$\Omega = \Omega_1 + \frac{1}{2}\left(\frac{(\lambda_1^2 - \lambda_2^2)}{\|\Omega_1\Omega_2\|^2} + 1\right)\overrightarrow{\Omega_1\Omega_2}, \ R^2 = \lambda_1^2 - \frac{1}{4}\frac{(\lambda_1^2 - \lambda_2^2 + \|\Omega_1\Omega_2\|^2)^2}{\|\Omega_1\Omega_2\|^2}.$$

Symmetrizing these formulae we get

$$\Omega = \frac{\Omega_1 + \Omega_2}{2} + \frac{(\lambda_1^2 - \lambda_2^2)}{2}\frac{\overrightarrow{\Omega_1\Omega_2}}{\|\Omega_1\Omega_2\|^2}, \ 4R^2 = 2(\lambda_1^2 + \lambda_2^2) - \|\Omega_1\Omega_2\|^2 - \frac{(\lambda_1^2 - \lambda_2^2)^2}{\|\Omega_1\Omega_2\|^2}. \tag{2}$$

Expressing the previous formulae with coordinates reveals unexpected algebraic properties which will be useful in the next section. Choosing an adapted frame, we can let A_1 be the origin, $N_1 = (0,0,1)$, $N_2 = (l,m,n)$, and $A_2 = (0, y_2, z_2)$. It follows that

$$\|A_1A_2\|^2 = y_2^2 + z_2^2,\ \overrightarrow{N_1} \cdot \overrightarrow{A_1A_2} = z_2,\ \overrightarrow{N_2} \cdot \overrightarrow{A_2A_1} = my_2 + nz_2.$$

Then, simple formal computations which could be done by hand but are easier with a computer algebra system yields the following properties.

Lemma 2. *Using the above notation, the quantity*

$$S = 4\frac{\|\Omega_1\Omega_2\|^2}{\|A_1A_2\|^2} \cdot (\overrightarrow{N_1} \cdot \overrightarrow{A_1A_2}) \cdot (\overrightarrow{N_2} \cdot \overrightarrow{A_2A_1})$$

is a polynomial in the input parameters, namely

$$S = l^2({y_2}^2{z_2}^2 + {z_2}^4) + m^2(y_2^2 - z_2^2)^2 + 4\, mny_2z_2(y_2^2 - z_2^2) + 4\, n^2{y_2}^2{z_2}^2.$$

In addition, the following properties hold:

i) $4 \cdot S \cdot R^2 = \|A_1A_2\|^2(S + l^2y_2^2(y_2^2 + z_2^2))$.

iii) the coordinates of $S \cdot \overrightarrow{\Omega M}$ *are polynomials in the input parameters,*

ii) the quantity $(\overrightarrow{\Omega_1\Omega_2} \cdot \overrightarrow{\Omega M}) \cdot (\overrightarrow{N_1} \cdot \overrightarrow{A_1A_2}) \cdot (\overrightarrow{N_2} \cdot \overrightarrow{A_2A_1})$ *is a polynomial in the input parameters, namely*

$$X\,(l{y_2}^2z_2 + l{z_2}^3) + Y(-m{y_2}^2z_2 + m{z_2}^3 - 2ny_2{z_2}^2) + Z(m{y_2}^3 - my_2{z_2}^2 + 2n{y_2}^2z_2).$$

This latter quantity provides a reduced equation of the supporting plane of the circle $\mathcal{C}$.

Case (a_2)**.** Now, consider the case where Ω_1 is at infinity but not Ω_2. Then the unique circle solution of the fitting problem is the intersection of the plane normal to N_1 and the sphere at Ω_2, both passing through A_1. With the previously chosen notation for the frame and the coordinates, we now have $z_2 = 0$, $y_2 \neq 0$ and $m \neq 0$. A straightforward computation shows that the equation of the plane containing the circle is $Z = 0$ and in this plane the equation of the circle (multiplied by $4m^2$ which is nonzero) is

$$(2mX - ly_2)^2 + (2mY - y_2m)^2 = m^2y_2^2 + l^2y_2^2.$$

Thus, its center Ω and the square of its radius R are given by

$$\Omega = \left(\frac{-ly_2}{2m}, \frac{y_2}{2}, 0\right),\quad R^2 = \frac{y_2^2(l^2 + m^2)}{4m^2}.$$

We also notice that the quantity S defined in Lemma 2, case (a_1), specializes when $z_2 = 0$ to $S = m^2y_2^4$ which is nonzero. Actually, the specialization of the entire Lemma 2 is still valid so that it can always be applied when the 3D circle fitting problem admits a unique solution.

Therefore, case (a_2) does not require a specific treatment compared to the case (a_1) when dealing with explicit formulae.

Cases (b) **and** (c)**.** In case (b) there is no solution and this case can be detected with the following test : $z_2 = 0,\ y_2 \neq 0,$ and $m = 0,\ l \neq 0$. In case (c) there are infinitely many solutions and it can be detected with the following test : $m^2 = 2ln$ and $m \neq 0$; or $z_2 = 0$ and $l = m = 0$.

We notice that the stronger condition "$m \neq 0$ and $m^2 \neq 2ln$" excludes both cases (b) and (c) and only depends on the input normals (N_1 and N_2).

3 Tori passing through a minimal point set

A torus is defined as the set of points in the three-dimensional affine space $\mathbb{R}^3$ that are located at a fixed distance, called the *small radius* of the torus, of a given circle which is called the *skeletal circle* of the torus; the radius of this circle will be called the *big radius* of the torus and will be denoted by R, while the small radius will be denoted by r. Seven parameters are needed for a torus: six for its skeletal circle and an additional one for the small radius.

Below, we will prove our main theorem; i.e. we will solve the problem of computing the tori passing through three distinct points, two of them being oriented. These points form a minimal point set because they correspond to seven conditions. Thus, our input consists in three distinct points P_1, P_2, P_3 and two nonzero normal vectors N_1, N_2 that are attached to the points P_1, P_2.

Implicit equations of tori. The points of a torus are exactly those points that are at distance r, the small radius of the torus, to the skeletal circle of the torus, which is a 3D circle of center Ω and radius R. We denote by $\overrightarrow{N}$ a normal vector to the supporting plane of the skeletal circle; a unitary normal vector is given by $\overrightarrow{N}/\|\overrightarrow{N}\|$. The squared distance of a point M to the skeletal circle can be classically derived as

$$\left(\frac{\overrightarrow{N}}{\|\overrightarrow{N}\|}\cdot\overrightarrow{\Omega M}\right)^2+\left(\sqrt{\|\Omega M\|^2-\left(\frac{\overrightarrow{N}}{\|\overrightarrow{N}\|}\cdot\overrightarrow{\Omega M}\right)^2}-R\right)^2.$$

Expressing that this latter quantity must be equal to r^2 and squaring we obtain the following implicit equation of the torus:

$$\left(\|\Omega M\|^2+R^2-r^2\right)^2=4R^2\left(\|\Omega M\|^2-\left(\frac{\overrightarrow{N}}{\|\overrightarrow{N}\|}\cdot\overrightarrow{\Omega M}\right)^2\right);$$

hence:

$$\|\overrightarrow{N}\|^2\left(\|\Omega M\|^2+R^2-r^2\right)^2-4R^2\left(\|\overrightarrow{N}\|^2\cdot\|\Omega M\|^2-(\overrightarrow{N}\cdot\overrightarrow{\Omega M})^2\right)=0. \tag{3}$$

Proof of the theorem. First, by a change of coordinates we can assume that P_1 is at the origin with $N_1=(0,0,1)$, and also that $P_2=(0,y_2,z_2)$. We set $N_2=(l,m,n)$ and $P_3=(x_3,y_3,z_3)$.

We introduce two new quantities r_1 and r_2 and we consider the two points $A_1=P_1+r_1N_1$ and $A_2=P_2+r_2N_2$. These two points will belong to the skeletal circle of a torus, of small radius equal to $|r_1|$, that interpolates the two oriented points P_1 and P_2 providing

$${r_1}^2={r_2}^2\|N_2\|^2={r_2}^2(l^2+m^2+n^2). \tag{4}$$

Assume that we are in the first case of Proposition 1, i.e. the subcases (a_1) or (a_2), and so that there is a unique circle that interpolates the points A_1 and A_2 and which is normal to N_1 at A_1 and normal to N_2 at A_2. This circle is the skeletal circle of the interpolating torus. Taking again the notation of Section 2, we denote by $\Omega=(a,b,c)$ its center, by R its radius and by $\overrightarrow{N}:=(u,v,w)$ which is a normal vector to the supporting plane of this circle. From Equation (3), (1) and (2), the implicit equation of this interpolating torus can be written in terms of the space coordinates x,y,z and of the parameters y_2,z_2,l,m,n and r_1,r_2. It turns out that, thanks to the algebraic remarks summarized in Lemma 2 of the previous section, this equation can be factorized and replaced by a simplified polynomial that we will denote by E. To be more precise, assume that we are in the case (a_1) and define the following polynomial quantities:

$$P:=\|A_1A_2\|^2={r_2}^2l^2+m^2{r_2}^2+n^2{r_2}^2+2mr_2y_2-2nr_1r_2+2nr_2z_2+{r_1}^2-2r_1z_2+{y_2}^2+{z_2}^2,$$

$$Q_1:=\overrightarrow{N_1}.\overrightarrow{A_1A_2}=nr_2-r_1+z_2,\quad Q_2:=\overrightarrow{N_2}.\overrightarrow{A_2A_1}=l^2r_2+m^2r_2+n^2r_2+my_2-nr_1+nz_2.$$

Then, using the properties given in Lemma 2, straightforward symbolic computations show that the implicit equation of the torus (3) writes as

$$\frac{P}{Q_1^2 \cdot Q_2^2} \cdot E(x, y, z, x_2, y_2, l, m, n, r_1, r_2) \tag{5}$$

where E is a polynomial. Similar computations shows that this polynomial also encapsulates the case (a_2). The polynomial equation $E = 0$ is obviously of degree 4 in the variables x, y, z. It turns out that it is of degree 8 with respect to each of the variables l, m, of degree 6 with respect to each of the variables n, y_2, z_2, r_2 and of degree 7 with respect to the variable r_1.

Summarizing the above calculations, the tori that interpolate the three distinct points P_1, P_2, P_3, where P_1 are P_2 are oriented by N_1 and N_2 respectively, are given by the two algebraic equations

$$E(x_3, y_3, z_3, x_2, y_2, l, m, n, r_1, r_2) = 0$$

(the variables x, y, z have been substituted by the point P_3 in order to impose the interpolation condition at this point) and $U := {r_1}^2 - {r_2}^2(l^2 + m^2 + n^2) = 0$ deduced from the condition (4). Since the polynomial U is monic in r_1, we can perform an Euclidian division of E by U with respect to to variables r_1, which remainder is a polynomial in r_1 of degree at most 1, that we denote by $S_1 + r_1 S_2$. Then, the system of equations finally reduces to

$$S_1 + r_1 S_2 = 0, \quad {r_1}^2 - {r_2}^2(l^2 + m^2 + n^2) = 0 \tag{6}$$

where S_1 and S_2 are polynomials in $x_3, y_3, z_3,\ l, m, n, y_2, z_2, r_2$ of degree 4 and 3 respectively.

The elimination of r_1 from the two equations in the system (6) yields a polynomial in $x_3, y_3, z_3,\ l, m, n,$ y_2, z_2, r_2 which is of degree 8 with respect to the variable r_2. Since the quantities $x_3, y_3, z_3,\ l, m, n, y_2, z_2$ are input data, one can solve r_2 from this degree 8 univariate polynomial and then build a unique torus for each of these values. Thus, we have proved the claimed theorem.

For the sake of completeness, we mention that the above elimination of r_1 in (6) can also be written as two univariate degree 4 polynomials. Indeed, the second equation in (6) shows that $r_1 = \pm r_2\sqrt{l^2 + m^2 + n^2}$. Thus, this gives two cases and for each case r_1 can be substituted in the first equation of (6) to provide a degree 4 univariate polynomial in r_2.

To conclude this section, we mention that the upper bound of eight tori is reached on examples, even under the restriction of smooth tori. For instance, the following input data are interpolated by eight smooth tori, as show in Figure 2 : $P_1 = (0, 0, 0)$, $N_1 = (0, 0, 1)$, $P_2 = (0, -3.171791777, -2.369900007)$, $N_2 = (-3.736353882, -2.170588024, 0.215631583)$ and $P_3 = (-0.394882587, 3.246764454, -3.362188875)$. See also Figure 3 for another illustrative example.

4 Effective solving and experiments

In order to apply the previous method for the extraction of tori in 3D point clouds by means of RANSAC-based approaches, we need to devise an algorithm that is fast and adapted to approximate data and numerical computations with double precision since this is the standard of the current libraries dealing with surface reconstruction.

The algebraic strategy we developed aims at producing closed form formulae that can be stored and used efficiently to solve the interpolation problem. Thus, the degree 8 polynomial in the variable r_2 we found (or the two degree 4 polynomials) could be pre-computed and its coefficients could be stored. Then, for each particular instance this polynomial could be solved and the tori extracted. However, the coefficients of this polynomial are big polynomials and their evaluation in double precision with approximate data yield significant numerical instability. In order to overcome this difficulty, we use a matrix-based formulation of the elimination of the variable r_1 in the system (6). More precisely, we form the Sylvester matrix of these two

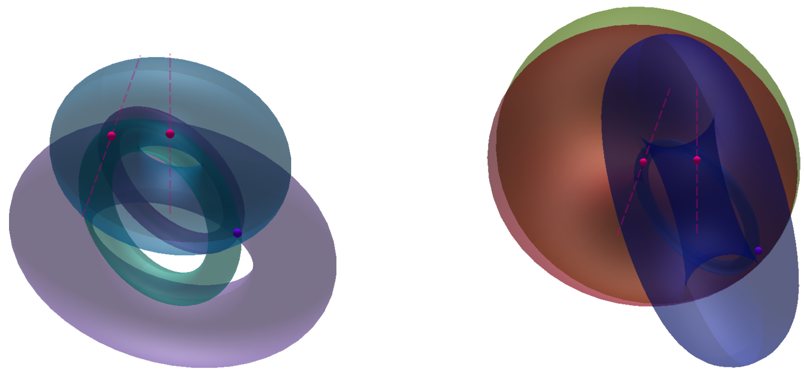

Figure 2: Eight tori that interpolate a minimal point set, separated into two groups of four torus in order to improve the visualization.

polynomials with respect to the variable r_1. This is a 3×3-matrix whose entries are polynomials of degree at most 4 in the variable r_2. Then, we compute its associated pencil of companion matrices A, B which are 12×12 matrices. For each particular instance, the generalized eigenvalues and eigenvectors of these matrices are computed to provide the couple of roots r_1, r_2 of this system (see [1] or [2] for more details). In practice, the matrices A and B are pre-computed and stored. It turns out that their entries are of smaller size and degree compared to the coefficients of the above degree 8 polynomial in r_2. This and the use of generalized eigenvalues yield a much more numerically stable algorithm.

A prototype of the interpolation algorithm we described above has been implemented in `C++`*, with the help of the computer algebra software `Maple` for computing the matrix-based closed formulae. We observed that the extraction of the tori from a random set of three points, two of them being oriented, is very fast and that its cost is almost constant independently of the point set, about 0.2ms. Contrary to the usual approaches (e.g. [3, 4, 7]), our method may output several tori for a given point set, up to eight tori. Nevertheless, we observed that less than four tori are obtained in 97% of cases in average. In practice, the reduction of the number of tori, typically to select only one or no torus, can also be done by taking into account the additional information of the normal at the third point P_3 if available. Indeed, it might be the case that this normal is expected to be in a certain cone of tolerance and hence the tori we computed can be filtered by comparing the orientation of P_3 for each torus with this cone.

5 Conclusion

This work continues our study of optimized basic shapes interpolation from the smallest possible set of mixed oriented points. We presented a point of view which combines geometric and algebraic aspects. The minimal (mixed) point set for defining a finite number of tori consists in two oriented points and a simple one, corresponding to seven numbers of freedom. We proposed an original analysis and an efficient algorithm for performing this extraction. Its main step consists of computing the generalized eigenvalues of a pre-computed pencil of matrices in closed-form.

References

[1] Laurent Busé, André Galligo, and Jiajun Zhang. Extraction of cylinders and cones from minimal point sets. *Graphical Models*, 86:1–12, 2016.

*ASurfExt C++ library :`https://gitlab.inria.fr/lbuse/ASurfExt/wikis/home`

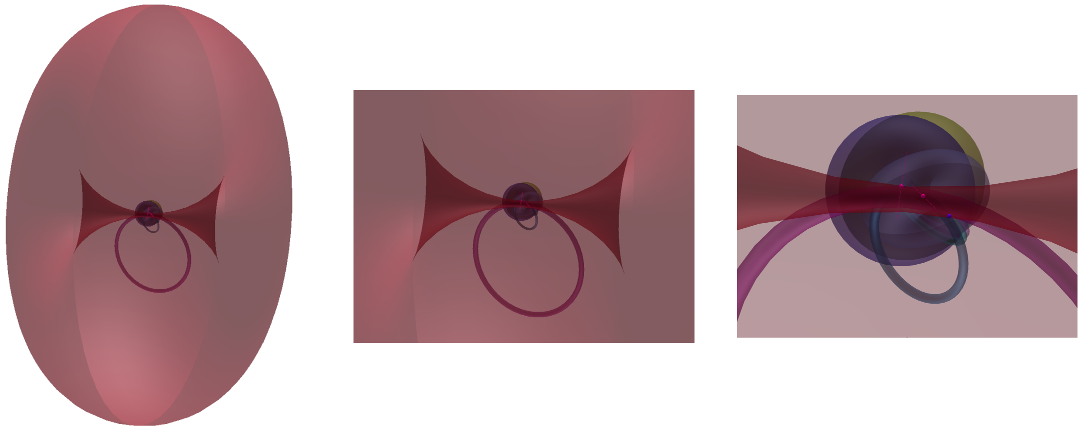

Figure 3: Eight smooth tori that interpolate a point set. The focus is increased from left to right in order to visualize all the eight tori.

[2] Laurent Busé, Houssam Khalil, and Bernard Mourrain. Resultant-based methods for plane curves intersection problems. In Victor G. Ganzha, Ernst W. Mayr, and Evgenii V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing (CASC)*, volume 3718, pages 75–92, Kalamata, Greece, September 2005. Springer Berlin / Heidelberg.

[3] Martin A. Fischler and Robert C. Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. *Commun. ACM*, 24(6):381–395, June 1981.

[4] John R. Kender and Rick Kjeldsen. On seeing spaghetti: A novel self-adjusting seven parameter hough space for analyzing flexible extruded objects. In *Proceedings of the 12th International Joint Conference on Artificial Intelligence - Volume 2*, IJCAI'91, pages 1271–1277, San Francisco, CA, USA, 1991. Morgan Kaufmann Publishers Inc.

[5] John R. Kender and Rick Kjeldsen. On seeing spaghetti: Self-adjusting piecewise toroidal recognition of flexible extruded objects. *IEEE Trans. Pattern Anal. Mach. Intell.*, 17(2):136–157, 1995.

[6] Gabor Lukács, Ralph Martin, and Dave Marshall. Faithful least-squares fitting of spheres, cylinders, cones and tori for reliable segmentation. In Hans Burkhardt and Bernd Neumann, editors, *Computer Vision — ECCV'98*, volume 1406 of *Lecture Notes in Computer Science*, pages 671–686. Springer Berlin Heidelberg, 1998.

[7] Ruwen Schnabel, Roland Wahl, and Reinhard Klein. Efficient ransac for point-cloud shape detection. *Computer Graphics Forum*, 26(2):214–226, June 2007.

[8] Zahra Toony, Denis Laurendeau, and Christian Gagné. Pgp2x: Principal geometric primitives parameters extraction. In *Proc. of the 10th International Conference on Computer Graphics Theory and Applications (GRAPP)*, 2015.